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A NON-HOMOGENEOUS ORBIT CLOSURE OF A DIAGONAL SUBGROUP

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Abstract. Let $G = \mathbf{SL}(n, \mathbf{R})$ with $n \geq 6$. We construct examples of lattices $\Gamma \subset G$, subgroups A of the diagonal group D and points $x \in G/\Gamma$ such that the closure of the orbit Ax is not homogeneous and such that the action of A does not factor through the action of a one-parameter non-unipotent group. This contradicts a conjecture of Margulis.

1. Introduction

1.1. **Topological rigidity and related questions.** Let G be a real Lie group, Γ a lattice in G, meaning a discrete subgroup of finite covolume, and A a closed connected subgroup. We are interested in the action of A on G/Γ by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [6] and [3] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset $X \subset G$ is said to be *homogeneous* if there exists a closed connected subgroup $H \subset G$ such that $X = Hx$ for some (and hence every) $x \in X$. Let us say that the action of A on G/Γ is *topologically rigid* if for any $x \in G/\Gamma$, the closure $\overline{Ax}$ of the orbit Ax is homogeneous.

The most basic example of a topologically rigid action is when $G = \mathbf{R}^n$, $\Gamma = \mathbf{Z}^n$, A any vector subspace of G. It turns out that the behavior of elements of A for the adjoint action on the Lie algebra $\mathfrak{g}$ of G plays a important role for our problem. Recall that an element $g \in G$ is said to be $\mathbf{Ad}$-unipotent if $\mathbf{Ad}(g)$ is unipotent, and $\mathbf{Ad}$-split over $\mathbf{R}$ if $\mathbf{Ad}(g)$ is diagonalizable over $\mathbf{R}$. If the closed, connected subgroup A of G is generated by $\mathbf{Ad}$-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of A is always topologically rigid, settling a conjecture due to Raghunathan.

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When A is generated by elements which are $\mathbf{Ad}$-split over $\mathbf{R}$, much less is known. Consider the model case of $G = \mathbf{SL}(n, \mathbf{R})$ and A the group of diagonal matrices with nonnegative entries. If $n = 2$, it is easy to produce non-homogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when A is a one-parameter subgroup of the diagonal group (see [6], 4.1). However, for A the full diagonal group, if $n \geq 3$, to the best of our knowledge, the only nontrivial example of a nonhomogeneous A-orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice Γ of $G = \mathbf{SL}(3, \mathbf{R})$ and a point $x \in G/\Gamma$ such that for the full diagonal group A, the orbit closure $\overline{Ax}$ is not homogeneous. Her construction was based on the following property of the lattice: there exists a $\gamma \in \Gamma \cap A$ such that the centralizer $C_G(\gamma)$ of γ is isomorphic to $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{R}^*$, and such that $C_G(\gamma) \cap \Gamma$ is, in this product decomposition and up to finite index, $\Gamma_0 \times \langle \gamma \rangle$, where Γ_0 is a lattice in $\mathbf{SL}(2, \mathbf{R})$ (see [4], [7]). Thus in this case the action of A on $C_G(\gamma)/C_G(\gamma) \cap \Gamma$ factors to the action of a 1-parameter non-unipotent subgroup on $\mathbf{SL}(2, \mathbf{R})/\Gamma_0$, which, as we saw, has many non-homogeneous orbits.

Rees' example shows that factor actions of 1-parameter non-$\mathbf{Ad}$-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, conjecture 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

Conjecture 1. *Let G be a connected Lie group, Γ a lattice in G, and A a closed, connected subgroup of G generated by $\mathbf{Ad}$-split over $\mathbf{R}$ elements. Then for any $x \in G/\Gamma$, one of the following holds :*

(a) *$\overline{Ax}$ is homogeneous, or*
(b) *There exists a closed connected subgroup F of G and a continuous epimorphism ϕ of F onto a Lie group L such that*
   – *$A \subset F$,*
   – *Fx is closed in G/Γ ,*
   – *$\phi(F_x)$ is closed in L, where F_x denotes the stabilizer $\{g \in F | gx = x\}$,*
   – *$\phi(A)$ is a one-parameter subgroup of L containing no nontrivial $\mathbf{Ad}_L$-unipotent elements.*

A first step toward this conjecture has been done by Lindenstrauss and Weiss [7], who proved that in the case $G = \mathbf{SL}(n, \mathbf{R})$ and A the full diagonal group, if the closure of a A-orbit contains a compact A-orbit that satisfy some irrationality conditions, then this closure is homogeneous. See also [15]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved that if moreover $\Gamma = \mathbf{SL}(n, \mathbf{Z})$, then the set of bounded A-orbits has Hausdorff dimension $n - 1$ [3, Theorem 10.2].

1.2. **Statement of the results.** In this article we exhibit some counterexamples to the above conjecture when $G = \mathbf{SL}(n, \mathbf{R})$ for $n \geq 6$ and A is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let D be the diagonal subgroup of G; note that D has dimension $n-1$. Our main result is:

Theorem 1. *Assume $n \geq 6$.*

(1) *There exists a $(n-3)$ dimensional closed and connected subgroup A of D, and a point $x \in \mathbf{SL}(n,\mathbf{R})/\mathbf{SL}(n,\mathbf{Z})$ such that the closure of the A-orbit of x satisfies neither condition* (a) *nor condition* (b) *of the conjecture.*

(2) *There exists a lattice Γ of $\mathbf{SL}(n,\mathbf{R})$, a $(n-2)$ dimensional closed and connected subgroup A of D and a point $x \in \mathbf{SL}(n,\mathbf{R})/\Gamma$ such that the closure of the A-orbit of x satisfies neither condition* (a) *nor condition* (b) *of the conjecture.*

It will be clear from the proofs that these examples however satisfy a third condition:

(c) *There exists a closed connected subgroup F of G and two continuous epimorphisms ϕ_1, ϕ_2 of F onto Lie groups L_1, L_2 such that*
- *$A \subset F$,*
- *Fx is closed in G/Γ ,*
- *For $i = 1, 2$, $\phi_i(F_x)$ is closed in L_i,*
- *$(\phi_1, \phi_2) : F \to L_1 \times L_2$ is surjective*
- *$(\phi_1, \phi_2) : A \to \phi_1(A) \times \phi_2(A)$ is not surjective.*

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

1.3. **Toral endomorphisms.** To conclude this introduction, we would like to mention that the idea behind this construction can be also used to yield examples of 'non-homogeneous' orbits for diagonal toral endomorphisms.

Let $1 < p_1 < \cdots < p_q$, with $q \geq 2$, be integers generating a multiplicative non-lacunary semigroup of $\mathbf{Z}$ (that is, the $\mathbf{Q}$-subspace $\oplus_{1\leq i\leq q}\mathbf{Q}\log(p_i)$ has dimension at least 2). We consider the abelian semigroup Ω of endomorphisms of the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ generated by the maps $z \mapsto p_i z \bmod \mathbf{Z}^n$, $1 \leq i \leq q$.

In the one-dimensional situation, described by Furstenberg [5], every Ω-orbit is finite or dense. If $n \geq 2$, Berend [1] showed that minimal sets are the finite orbits of rational points, but there are others obvious closed Ω-invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integer Hausdorff dimension.

Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if $q = n = 2$, and if $\alpha \in T^1$ is irrational with non-dense p_1-orbit, the orbit closure of the point $(\alpha, 1/p_2) \in T^2$ is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).

With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for $n \geq 2q$, as the following example shows.

Theorem 2. *Let N be an integer greater than $q\frac{\log p_q}{\log p_1}$, and let z be the point in the $2q$-dimensional torus T^{2q} defined by the coordinates modulo 1:*

$$z = (z_1, \ldots, z_{2q}) = \left(\sum_{k \geq 1} p_1^{-N^{2k}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k+1}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k+1}} \right).$$

Then the point $z \in T^{2q}$ is not contained in any rational affine subspace, but its orbit Ωz is not dense.

The proof of Theorem 2 will be the subject of Section 4.

2. Sketch of proof of Theorem 1

2.1. **The direct product setup.** We now describe how these examples are built. Choose two integers $n_1 \geq 3$, $n_2 \geq 3$, such that $n_1 + n_2 = n$. For $i = 1, 2$, let Γ_i be a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$.

Let g_i be an element of G_i such that $g_i \Gamma_i g_i^{-1}$ intersects the diagonal subgroup D_i of $\mathbf{SL}(n_i, \mathbf{R})$ in a lattice, in other words $g_i \Gamma_i$ has a compact D_i-orbit; such elements exist, see [11]. In fact, we will need an additional assumption on g_i, namely that the tori $g_i^{-1} D_i g_i$ are *irreducible over* $\mathbf{Q}$. The precise definition of this property and the proof of the existence of such a g_i, a consequence of a theorem of Prasad and Rapinchuk [12, Theorem 1], will be the subject of Section 3.1.

Let $\pi_i : G_i \to G_i/\Gamma_i$ be the canonical quotient map. Define for $i = 1, 2$:

$$y_i = \pi_i \left(\begin{bmatrix} 1 & 0 & \ldots & 0 & 1 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} g_i \right).$$

The D_i-orbit of y_i is dense, by the following argument. It is easily seen that the closure of $D_i y_i$ contains the compact D_i-orbit $\mathcal{T}_i = \pi_i(D_i g_i)$. The $\mathbf{Q}$-irreducibility of $\mathcal{T}_i$ is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [7, Theorem 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group $H_i < G_i$ such that $H_i y_i = \overline{D_i y_i}$. Again because of **Q**-irreducibility, the group H_i is necessarily the full group, i.e. $H_i = G_i$ (proof of Lemma 3.2) [1].

Let A_1 be the $(n-3)$ dimensional subgroup of $G_1 \times G_2$ given by:

$$
\begin{aligned}
A_1 = \Big\{ & (diag(a_1,..,a_{n_1}), diag(b_1,..,b_{n_2})) \ : \\
& \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1 b_1}{a_{n_1} b_{n_2}} = 1,\ a_i > 0, b_j > 0 \Big\}.
\end{aligned} \tag{1}
$$

Then the A_1-orbit of (y_1, y_2) is not dense in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ (Lemma 3.3), but $G_1 \times G_2$ is the smallest closed connected subgroup F of $G_1 \times G_2$ such that $\overline{A_1(y_1, y_2)} \subset F(y_1, y_2)$ (Lemma 3.7).

This yields a counterexample to Conjecture 1 which can be summarized as follows:

Proposition 1. *For $i = 1, 2$, let $n_i \geq 3$ and Γ_i be a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$. For A_1, y_1, y_2 depicted as above, the A_1-orbit of (y_1, y_2) in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ satisfies neither condition (a) nor condition (b) of Conjecture 1.*

2.2. **Theorem 1, part (1).** In order to obtain the first part of Theorem 1, choose $\Gamma_i = \mathbf{SL}(n_i, \mathbf{Z})$, $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ and consider the embedding of $G_1 \times G_2$ in G, where matrices are written in blocks:

$$
\Psi \ : \ (M_{n_1,n_1}, N_{n_2,n_2}) \mapsto \begin{bmatrix} M_{n_1,n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & N_{n_2,n_2} \end{bmatrix}. \tag{2}
$$

This embedding gives rise to an embedding $\overline{\Psi}$ of $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ into G/Γ. Let y_1, y_2 be two points as above, let $x = \overline{\Psi}(y_1, y_2)$ and take $A = \Psi(A_1)$. We claim that this point x and this group A satisfy Theorem 1, part (1). In fact, since the image of $\overline{\Psi}$ is a closed connected A-invariant subset of $\mathbf{SL}(n, \mathbf{R})/\mathbf{SL}(n, \mathbf{Z})$, everything takes place in this direct product.

2.3. **Theorem 1, part (2).** The second part of Theorem 1 is obtained as follows. Let σ be the nontrivial field automorphism of the quadratic extension $\mathbf{Q}(\sqrt[4]{2})/\mathbf{Q}(\sqrt{2})$. Consider for any $m \geq 1$:

$$
\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma) = \left\{ M \in \mathbf{SL}(m, \mathbf{Z}[\sqrt[4]{2}]) \ : \ ({}^t M^\sigma) M = I_m \right\}.
$$

Then $\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ is a lattice in $\mathbf{SL}(m, \mathbf{R})$, as will be proved in Section 3.5 (see [4, Appendix] for $m = 3$). Define for $i = 1, 2$, $\Gamma_i = \mathbf{SU}(n_i, \mathbf{Z}[\sqrt[4]{2}], \sigma)$, and $\Gamma = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$. Now consider the map:

$$
\varphi : G_1 \times G_2 \times \mathbf{R} \to G,
$$

[1] The reader only interested in the case $n = 6$ and $\Gamma = \mathbf{SL}(6, \mathbf{Z})$ might note that when $\Gamma_1 = \Gamma_2 = \mathbf{SL}(3, \mathbf{Z})$, [7, Corollary 1.4] can be used directly in the proof of Lemma 3.2; then the notion of **Q**-irreducibility becomes unnecessary, and the entire Section 3.1 can be skipped.

$$(X,Y,t) \mapsto \begin{bmatrix} e^{n_2 t}X & 0 \\ 0 & e^{-n_1 t}Y \end{bmatrix}.$$

Define M to be the image of φ. This time, φ factors into a finite covering $\overline{\varphi}$ of homogeneous spaces:

$$\overline{\varphi} : G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log\alpha)\mathbf{Z} \to M/M \cap \Gamma \subset G/\Gamma,$$

where $\alpha = (3+2\sqrt{2}) + \sqrt[4]{2}(2+2\sqrt{2})$ satisfies $\alpha^{-1} = \sigma(\alpha)$. Consider the points y_i constructed above, and let $x = \overline{\varphi}(y_1, y_2, 0)$. Choose:

$$A = \left\{ diag(a_1, .., a_n) \mid \prod_{i=1}^{n} a_i = \frac{a_1 a_{n_1+1}}{a_{n_1} a_n} = 1,\ a_i > 0 \right\} \subset \mathbf{SL}(n, \mathbf{R}).$$

We claim that this lattice Γ, this point x and this group A satisfy Theorem 1, part (2). What happens here is that the A-orbit of x is a circle bundle over an A_1-orbit (up to the finite cover $\overline{\varphi}$), like in Rees' example.

3. Proof of Theorem 1

3.1. **Q-irreducible tori.** Fix $i \in \{1,2\}$. Recall that Γ_i is a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$. Since $n_i \geq 3$, by Margulis's arithmeticity Theorem [16, Theorem 6.1.2], there exists a semisimple algebraic $\mathbf{Q}$-group $\mathbf{H}_i$ and a surjective homomorphism θ from the connected component of identity of the real points of this group $\mathbf{H}_i^0(\mathbf{R})$ to $\mathbf{SL}(n_i, \mathbf{R})$, with compact kernel, such that $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$ is commensurable with Γ_i.

Following Prasad and Rapinchuk, we say that a $\mathbf{Q}$-torus $\mathbf{T} \subset \mathbf{H}_i$ is $\mathbf{Q}$-irreducible if it does not contain any proper subtorus defined over $\mathbf{Q}$. By [12, Theorem 1,(ii)], there exists a maximal $\mathbf{Q}$-anisotropic $\mathbf{Q}$-torus $\mathbf{T}_i \subset \mathbf{H}_i$, which is $\mathbf{Q}$-irreducible. Because any two maximal $\mathbf{R}$-tori of $\mathbf{SL}(n_i, \mathbf{R})$ are $\mathbf{R}$-conjugate, there exists $g_i \in G_i$ such that $\theta(\mathbf{T}_i^0(\mathbf{R})) = g_i^{-1} D_i g_i$. The subgroup $\mathbf{T}_i(\mathbf{Z})$ is a cocompact lattice in $\mathbf{T}_i(\mathbf{R})$ since $\mathbf{T}_i$ is $\mathbf{Q}$-anisotropic [2, Theorem 8.4 and Definition 10.5]. Because $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$ and Γ_i are commensurable and θ has compact kernel, it follows that both $\Gamma_i \cap g_i^{-1} D_i g_i$ and $\theta(\mathbf{T}_i^0(\mathbf{Z})) \cap \Gamma_i \cap g_i^{-1} D_i g_i$ are also cocompact lattices in $g_i^{-1} D_i g_i$. The resulting topological torus $\pi_i(D_i g_i) \subset G_i/\Gamma_i$ will be denoted $\mathcal{T}_i$. Write $z_i = \pi_i(g_i)$, so that $\mathcal{T}_i = D_i z_i$.

For every $1 \leq k < l \leq n_i$, define as in [7]:

$$N_{k,l}^{(i)} = \left\{ diag(a_1, .., a_{n_i}) \ : \ \prod_{s=1}^{n_i} a_s = 1,\ a_k = a_l,\ a_s > 0 \right\} \subset D_i,$$

Of interest to us amongst the consequences of $\mathbf{Q}$-irreducibility is the fact that an element of $\Gamma_i \cap g_i^{-1} D_i g_i$ lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

Lemma 3.1. *For every $1 \leq k < l \leq n_i$, and any closed connected subgroup L of positive dimension of $N_{k,l}^{(i)}$, the L-orbit of z_i is not compact.*

Proof. Assume the contrary, that is Lz_i is compact. This implies that $g_i^{-1}Lg_i \cap \Gamma_i$ is a uniform lattice in $g_i^{-1}Lg_i$, so $g_i^{-1}Lg_i \cap \theta(\mathbf{H}_i(\mathbf{Z}))$ is also a uniform lattice. Since L is nontrivial, there exists an element $\gamma \in \mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R})$ of infinite order, such that $g_i\theta(\gamma)g_i^{-1}$ is in L. Note that since θ has compact kernel, $\mathbf{T}_i(\mathbf{Z})$ is a lattice in $\theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$ and is then a subgroup of finite index in $\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}) \cap \theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$, so there exists $n > 0$ such that γ^n belongs to $\mathbf{T}_i(\mathbf{Z})$. Consider the representation:

$$\begin{aligned} \rho : \mathbf{H}_i^0(\mathbf{R}) &\to \mathbf{GL}(\mathfrak{sl}(n_i, \mathbf{R})), \\ x &\mapsto \mathbf{Ad}(g_i\theta(x)g_i^{-1}). \end{aligned}$$

Recall that $\chi(diag(a_1, .., a_{n_1})) = a_k/a_l$ is a weight of $\mathbf{Ad}$ with respect to D_i, so χ is a weight of ρ with respect to $\mathbf{T}_i$. By [12, Proposition 1, (iii)], the $\mathbf{Q}$-irreducibility of $\mathbf{T}_i$ implies that $\chi(\gamma^n) \neq 1$, but this contradicts the fact that $\theta(\gamma^n) \in g_i^{-1}N_{k,l}^{(i)}g_i$. □

3.2. **Contraction and expansion.** For real s, denote by $a_i(s)$ the following $n_i \times n_i$-matrix:

$$a_i(s) = diag(e^{s/2}, \underbrace{1, \ldots, 1}_{n_i-2 \text{ times}}, e^{-s/2}),$$

and write simply N_i for $N_{1,n_i}^{(i)}$. Write also:

$$h_i(t) = \begin{bmatrix} 1 & 0 & \ldots & 0 & t \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix}.$$

Then the following commutation relation holds:

$$a_i(s)h_i(t) = h_i(e^s t)a_i(s),$$

that is the direction h_i is expanded for positive s; note that both h_i and a_i commute with elements of N_i. It is easy to check from Equation (1) that

$$A_1 = \{(a_1(s)d_1, a_2(-s)d_2) \ : \ s \in \mathbf{R},\ d_i \in N_i,\ i = 1, 2\}.$$

Recall that $y_i = h_i(1)z_i$.

Lemma 3.2. (1) *If $s \leq 0$, for any $d \in N_i$ the point $a_i(s)dy_i$ lies in the compact set $K_i = h_i([0,1])\mathcal{T}_i$.*
(2) *The D_i-orbit of y_i is dense in G_i/Γ_i.*
(3) *The set $\{a_i(s)dy_i \ : \ s \geq 0,\ d \in N_i\}$ is dense in G_i/Γ_i.*

Proof. The first statement is clear from the commutation relation. It also implies that D_iy_i contains the compact torus $\mathcal{T}_i$ in its closure.

To prove the second point, we rely heavily on the paper of Lindenstrauss and Weiss. [7, Theorem 1.1] applies here, since the hypotheses of their Theorem is precisely the conclusion of Lemma 3.1 for $L = N_{k,l}^{(i)}$. So the following holds: there exists a reductive subgroup H_i, containing D_i, such that $\overline{D_i y_i} = H_i y_i$, and $H_i \cap \Gamma_i$ is a lattice in H_i. Write $L = D_i \cap C_{G_i}(H_i)$.

Since $D_i y_i$ is not closed, $H_i \neq D_i$, so there exists a nontrivial root relatively to D_i for the Adjoint representation of H_i on its Lie algebra, which is a subalgebra of $\mathfrak{sl}(n_i, \mathbf{R})$. Thus there exist k, l such that $L \subset N_{k,l}^{(i)}$. By [7, step 4.1 of Lemma 4.2], Lz_i is compact, so by Lemma 3.1, L is trivial. By [7, Proposition 3.1], H_i is the connected component of the identity of $C_{G_i}(L)$, so $H_i = G_i$, as desired.

The third claim follows from the first and second claim together with the fact that K_i has empty interior. □

3.3. Topological properties of the A_1-orbit.

Lemma 3.3. *The A_1-orbit of (y_1, y_2) is not dense in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$.*

Proof. Consider the open set $U = K_1^c \times K_2^c$. We claim that the A_1-orbit of (y_1, y_2) does not intersect U. Indeed, if $(a_1(s)d_1, a_2(-s)d_2) \in A_1$ with $s \in \mathbf{R}$ and $d_i \in N_i$, the previous Lemma implies that if $s \geq 0$, $a_2(-s)d_2y_2 \in K_2$, and if $s \leq 0$, $a_1(s)d_1y_1 \in K_1$. □

The following elementary result will be useful:

Lemma 3.4. *Let $p_i : G_1 \times G_2 \to G_i$ be the first (resp. second) coordinate morphism. If $F \subset G_1 \times G_2$ is a subgroup such that $p_i(F) = G_i$ for $i = 1, 2$, and $A_1 \subset F$, then $F = G_1 \times G_2$.*

Proof. Let $F_1 = Ker(p_1) \cap F$. Since F_1 is normal in F, $p_2(F_1)$ is normal in $p_2(F) = G_2$. Note that $N_2 \subset p_2(A_1 \cap Ker(p_1)) \subset p_2(F_1)$ is not finite, and that G_2 is almost simple, consequently the normal subgroup $p_2(F_1)$ of G_2 is equal to G_2. Let $(a, b) \in G_1 \times G_2$, by assumption there exists $f \in F$ such that $p_1(f) = a$. Let $f_1 \in F_1$ be such that $p_2(f_1) = bp_2(f)^{-1}$, then $(a, b) = f_1 f \in F$. □

We will have to apply several times the two following well-known Lemmas:

Lemma 3.5. *Let L be a Lie group, $\Lambda \subset L$ a lattice, M, N two closed, connected subgroups of L, such that for some $w \in L/\Lambda$, Mw and Nw are closed. Then $(M \cap N)w$ is closed.*

Proof. This is a weaker form of [14, Lemma 2.2]. □

Lemma 3.6. *Let L be a connected Lie group, $\Lambda \subset L$ a discrete subgroup, M, N two subgroups of L, such that M is closed and connected, and N is a countable union of closed sets. For any $w \in L/\Lambda$, if $Mw \subset Nw$, then $M \subset N$.*

Proof. Up to changing Λ by one of its conjugate in L, one can assume that $w = \Lambda \in L/\Lambda$. By assumption, $M\Lambda \subset N\Lambda$ so $M \subset N\Lambda \subset L$. Recall that M

is closed, that Λ is countable, and that N is a countable union of closed sets, so Baire's category Theorem applies, and there exists $\lambda \in \Lambda$ and an open set U of M such that $U \subset N\lambda$, so $UU^{-1} \subset N$. Since M is a connected subgroup, UU^{-1} generates M, so $M \subset N$. □

The following lemma will be useful both for proving that the closure of $A_1(y_1, y_2)$ is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

Lemma 3.7. *Let F be a closed connected subgroup of $G_1 \times G_2$ such that $F(y_1, y_2)$ contains the closure of $A_1(y_1, y_2)$. Then $F = G_1 \times G_2$.*

Proof. By Lemma 3.2, the set of first coordinates of the set

$$\{(a(s)d_1y_1, a(-s)d_2y_2) : s \geq 0, d_i \in N_i\},$$

is dense in G_1/Γ_1 and the second coordinates lies in the compact set K_2, so the closure of $A_1(y_1, y_2)$ contains points of arbitrary first coordinate with their second coordinate in K_2. Consequently, the set of first coordinates of $F(y_1, y_2)$ is the whole G_1/Γ_1, and similarly for the set of second coordinates. For $i = 1, 2$, Lemma 3.6 now applies to $L = M = G_i$, $\Lambda = \Gamma_i$, $N = p_i(F)$, which is a countable union of closed sets because $G_1 \times G_2$ is σ-compact, and $w = y_i$, and so $p_i(F) = G_i$.

In order to apply Lemma 3.4 and finish the proof, we have to show that $A_1 \subset F$. Again, this follows from a direct application of Lemma 3.6 to $L = G_1 \times G_2$, $\Lambda = \Gamma_1 \times \Gamma_2$, $M = A_1$, $N = F$, $w = (y_1, y_2)$.

□

3.4. **Proof of Theorem 1, part (1).** We now proceed to proving Theorem 1, part (1). The proof of Proposition 1 is similar and is omitted.

Recall that in this case, we fixed $A = \Psi(A_1)$ and $x = \overline{\Psi}(y_1, y_2)$.

Assume $\overline{Ax}$ is homogeneous, that is $\overline{Ax} = Fx$ for a closed connected subgroup F of G. Since $Ax \subset \overline{\Psi}(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$, which is closed in G/Γ, Lemma 3.6 imply that $F \subset \Psi(G_1 \times G_2)$. By Lemma 3.7, $F = \Psi(G_1 \times G_2)$, so $Fx = G/\Gamma$ and Ax is dense in $\overline{\Psi}(G_1 \times G_2)$, which is a contradiction.

Now assume $\overline{Ax}$ fibers over the orbit of a one-parameter subgroup. Let F be a closed connected subgroup, L a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$, we have $A \subset F'$. By Lemma 3.5, $F'x$ is closed in $Fx \cap \overline{\Psi}(G_1 \times G_2)$, so is closed in G/Γ. By Lemma 3.7 , $F' = \Psi(G_1 \times G_2)$ necessarily. Let $H = Ker(\phi \circ \Psi) \subset G_1 \times G_2$, so $A_1/(A_1 \cap H)$ is a one-parameter group by assumption (b).

The subgroup H is a normal subgroup of the semisimple group $G_1 \times G_2$, which has only four kind of normal subgroups : finite, $G_1 \times G_2$, $G_1 \times$finite and finite$\times G_2$. None of these possible normal subgroups have the property that they intersect A_1 in a codimension 1 subgroup, so this is a contradiction.

3.5. **The arithmetic lattice.** Here we prove that $\mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ is a lattice in $\mathbf{SL}(n, \mathbf{R})$. Let P, Q be the polynomials with coefficients in $\mathbf{Q}(\sqrt{2})$ such that for any $X, Y \in M_n(\mathbf{C})$

$$\det(X + \sqrt[4]{2}Y) = P(X,Y) + \sqrt[4]{2}Q(X,Y).$$

For an integral domain $\mathbf{A} \subset \mathbf{C}$, consider the set of pairs of matrices:

$$\mathbf{G}(\mathbf{A}) = \{(X,Y) \in M_n(\mathbf{A})^2 \ : {}^tXX - \sqrt{2}{}^tYY = I_n,\ {}^tXY - {}^tYX = 0,$$
$$P(X,Y) = 1,\ Q(X,Y) = 0\},$$

which implies that $({}^tX - \sqrt[4]{2}{}^tY)(X + \sqrt[4]{2}Y) = I_n$ and $\det(X + \sqrt[4]{2}) = 1$ for all $(X,Y) \in \mathbf{G}(\mathbf{A})$. Endow $\mathbf{G}(\mathbf{A})$ with the multiplication given by

$$(X,Y)(X',Y') = (XX' + \sqrt{2}YY', XY' + YX'),$$

which is such that the map $\phi : \mathbf{G}(\mathbf{A}) \to \mathbf{SL}(n, \mathbf{C})$, $(X,Y) \mapsto X + \sqrt[4]{2}Y$ is a morphism. With this structure, $\mathbf{G}$ is an algebraic group, which is clearly defined over $\mathbf{Q}(\sqrt{2})$. Let τ be the nontrivial field automorphism of $\mathbf{Q}(\sqrt{2})/\mathbf{Q}$, it can be checked that the map ϕ is an isomorphism between $\mathbf{G}(\mathbf{R})$ and $\mathbf{SL}(n, \mathbf{R})$, and that moreover $\phi' : \mathbf{G}^\tau(\mathbf{R}) \to \mathbf{SL}(n, \mathbf{C})$, $(X,Y) \mapsto X + i\sqrt[4]{2}Y$ is an isomorphism onto $\mathbf{SU}(n)$. Let $\mathbf{H} = Res_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}}\mathbf{G} = \mathbf{G} \times \mathbf{G}^\tau$. Then $\mathbf{H}$ is defined over $\mathbf{Q}$ (see for example [16, 6.1.3], for definition and properties of the restriction of scalars functor). It follows from a Theorem of Borel and Harish-Chandra [16, Theorem 3.1.7] that $\mathbf{H}(\mathbf{Z})$ is a lattice in $\mathbf{H}(\mathbf{R})$. Since $\mathbf{SU}(n)$ is compact, it follows that the projection of $\mathbf{H}(\mathbf{Z})$ onto the first factor of $\mathbf{G}(\mathbf{R}) \times \mathbf{G}^\tau(\mathbf{R})$ is again a lattice. Using the isomorphism between $\mathbf{G}(\mathbf{R})$ and $\mathbf{SL}(n, \mathbf{R})$, this projection can be identified with

$$\mathbf{G}(\mathbf{Z}[\sqrt{2}]) = \mathbf{SU}(n, \mathbf{Z}[\sqrt{2}] + \sqrt[4]{2}\mathbf{Z}[\sqrt{2}], \sigma) = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma).$$

3.6. **Proof of Theorem 1, part (2).** Note that, as stated implicitely in Section 2.3,

$$\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}) \subset \Gamma \cap M,$$

so $\Gamma \cap M$ is a lattice in M, and $M/(M \cap \Gamma)$ is a closed, A-invariant subset of G/Γ. Notice also that the map Ψ defined by Equation (2) defines an embedding $\overline{\Psi} : G_1 \times G_2/\Gamma_1 \times \Gamma_2 \to G/\Gamma$.

Assume $\overline{Ax}$ is homogeneous, that is $\overline{Ax} = Fx$ for a closed connected subgroup F of G. Since $Ax \subset M/(M \cap \Gamma)$, which is closed in G/Γ, Lemma 3.6 applied twice gives that $A \subset F \subset M$. Let $F' = F \cap \Psi(G_1 \times G_2)$, again by Lemma 3.5, $F'x$ is a closed subset of $Im(\overline{\Psi})$. Since $A_1 \subset F'$, $\Psi(A_1)x \subset F'x$ and Lemma 3.7 implies that $F' = \Psi(G_1 \times G_2)$. Since A contains $\varphi(e,e,t)$ for all $t \in \mathbf{R}$, we have $M = AF' \subset F$ so $F = M$ necessarily.

By Lemma 3.3, the A_1-orbit of (y_1, y_2) is not dense; the topological transitivity of the action of A_1 on $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ implies that moreover the closure of this orbit has empty interior. Thus, the $A_1 \times \mathbf{R}$-orbit of $(y_1, y_2, 0)$ is also nowhere dense in $G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}$. The map $\overline{\varphi}$ being a finite covering, the A-orbit of x is nowhere dense. This is a contradiction with $F = M$.

Now assume $\overline{Ax}$ fibers over the orbit of a one-parameter non-$\mathbf{Ad}$-unipotent subgroup. Let F be a closed connected subgroup, L a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$ and $F'' = F \cap M$, we have $A_1 \subset F'$ and $A \subset F''$. Similarly, $F'x$ and $F''x$ are closed in G/Γ. Again, by Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily, and like before, $AF' \subset F'' \subset M$ so $F'' = M$.

Let $H = Ker(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbf{R}$, so $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$ is a one-parameter group. This time, possibilities for the closed normal subgroup H are: finite $\times \Lambda$, $G_1 \times G_2 \times \Lambda$, $G_1 \times$ finite $\times \Lambda$ and finite $\times G_2 \times \Lambda$, where Λ is a closed subgroup of $\mathbf{R}$. Of all these possibilities, only $G_1 \times G_2 \times \Lambda$, where Λ is discrete, has the required property that $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$ is a one-parameter group. This proves that $\Psi(G_1 \times G_2) \subset Ker(\phi)$, so $F \subset N_G(\Psi(G_1 \times G_2))$. However, the normalizer of $\Psi(G_1 \times G_2)$ in G is the group of block matrices having for connected component of the identity the group M. So by connectedness of F, $F \subset M$, and since $M = F'' \subset F$, we have $F = M$. Thus $L = F/Ker(\phi) = \mathbf{R}/\Lambda$ is abelian, and a fortiori every element of L is unipotent; this contradicts (b).

4. Proof of Theorem 2

The proof of Theorem 2 is divided in two independent lemmas.

Lemma 4.1. *The family $(z_1, \ldots, z_{2q}, 1)$ is linearly independent over $\mathbf{Q}$.*

Proof. Consider a linear combination:

$$\sum_{i=1}^{q} a_i z_i + b_i z_{i+q} = c.$$

We can assume that a_i, b_i and c are integers. Let $k_0 \geq 1$, write

$$(3) \quad \left(\prod_{i=1}^{q} p_i\right)^{N^{2k_0+1}} \left(\sum_{i=1}^{q} \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c\right) = - \left(\prod_{i=1}^{q} p_i\right)^{N^{2k_0+1}} \left(\sum_{i=1}^{q} \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}}\right).$$

It is clear the left hand side is an integer. Since $1 < p_1 < \cdots < p_q$, the right hand side is less in absolute value than

$$p_q^{qN^{2k_0+1}} 2q \sup_i(|a_i|, |b_i|) \sum_{k\geq 0} \left(p_1^{-N^{2k_0+2}}\right)^{N^{2k}}$$
$$\leq 4q \sup_i(|a_i|, |b_i|) p_q^{qN^{2k_0+1}} p_1^{-N^{2k_0+2}}$$
$$\leq 4q \sup_i(|a_i|, |b_i|) \exp(N^{2k_0+1}(q \log p_q - N \log p_1)).$$

Since $N > q\frac{\log(p_q)}{\log(p_1)}$, the last expression tends to zero . This proves the right-hand side of (3) is zero for large enough k_0, so for all large k,

$$\sum_{i=1}^{q} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.$$

The p_i being distincts, this implies that for $i \in \{1, .., q\}$, $a_i = b_i = 0$. □

The following Lemma implies easily that the orbit of z under Ω cannot be dense.

Lemma 4.2. *For all $\epsilon > 0$, there exists $L > 0$, such that for all $n_1, .., n_q \geq 0$ with $\sum_{i=1}^q n_i \geq L$, there exists $j \in \{1, \ldots, 2q\}$ such that $p_1^{n_1} \cdots p_q^{n_q} z_j$ lies in the interval $[0, \epsilon]$ modulo 1.*

Proof. Let $s \in \{1, \ldots, q\}$ such that for all $r \in \{1, \ldots, q\}$, $p_s^{n_s} \geq p_r^{n_r}$. Let k_0 be the integer part of $\log(n_s)/2\log(N)$, then either $N^{2k_0} \leq n_s \leq N^{2k_0+1}$, or $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$. In the first case, take $j = s$, then:

$$p_1^{n_1} \cdots p_q^{n_q} z_j = p_1^{n_1} \cdots p_q^{n_q} \sum_{k\geq 1} p_s^{-N^{2k}} = p_1^{n_1} \cdots p_q^{n_q} \sum_{k\geq k_0+1} p_s^{-N^{2k}} \bmod 1.$$

We have

$$\sum_{k\geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{-N^{2k_0+2}},$$

so, using the fact that for all $r \in \{1, \ldots, q\}$, $p_r^{n_r} \leq p_s^{n_s} \leq p_s^{N^{2k_0+1}}$, we obtain:

$$p_1^{n_1} \cdots p_q^{n_q} \sum_{k\geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{qN^{2k_0+1}-N^{2k_0+2}} \leq 2p_s^{N^{2k_0+1}(q-N)},$$

but by hypothesis we have $N > q\frac{\log(p_q)}{\log(p_1)} > q$, so the preceding bound is small whenever k_0 is large. Because of the definition of k_0, we have

$$k_0 \geq \frac{\log \frac{\sum_{i=1}^q n_i \log p_i}{q \log p_q}}{2 \log N} \geq \frac{\log \frac{L \log p_1}{q \log p_q}}{2 \log N},$$

so k_0 is arbitrary large when L is large.

In the second case $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$, one can proceed similarly with $j = s + q$.

□

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